NORTH-HOLLAND

Perron-Frobenius Theory for Finite-Dimensional Spaces With a Hyperbolic Cone

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ABSTRACT

If X is a real n-dimensional space provided with a subnorm π, then the inequality $\xi \geqslant \pi(x)$ defines a so-called *hyperbolic cone* in $E = \mathbb{R} \oplus X$. In this case the Perron-Frobenius theory admits some special features. A relevant characterization of nonnegative operators in a matrix form is given first. Auxiliary information from spectral theory and from the geometry of subnormed spaces is collected as preparation.

1. A SPECTRAL THEORY OF ONE-DIMENSIONAL EXTENSIONS

Let X be a real n-dimensional space, $n < \infty$. We will deal with a one-dimensional extension of X,

$$E = \mathbb{R} \oplus X = \left\{ u : u = \begin{pmatrix} \xi \\ x \end{pmatrix}, \xi \in \mathbb{R}, x \in X \right\}.$$

The conjugate space E^* consisting of all linear functionals on E may be described in a similar form,

$$E^* = R \oplus X^* = \{ g : g = (\eta, f), \eta \in \mathbb{R}, f \in X^* \},$$

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and

$$g(u) = \eta\xi + f(x). \tag{1.1}$$

To get (1.1) for a given $g \in E^*$ it is sufficient to write

$$u = \begin{pmatrix} \xi \\ x \end{pmatrix} = \xi\begin{pmatrix} 1 \\ \overline{0} \end{pmatrix} + \begin{pmatrix} 0 \\ x \end{pmatrix}. \tag{1.2}$$

Then (1.1) is valid with

$$\eta = g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \qquad f(x) = g\left(\begin{pmatrix} 0 \\ x \end{pmatrix}\right). \tag{1.3}$$

It is easy to see that the formulas (1.3) define an isomorphism $E^* \to \mathbb{R} \oplus X^*$. We will keep it as a canonical isomorphism to identify E^* and $\mathbb{R} \oplus X^*$.

PROPOSITION 1.1. *The space $L(E)$ of all linear operators in E can be identified with the space of all matrices*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \tag{1.4}$$

where $\eta \in \mathbb{R}$, $z \in X$, $T \in L(X)$, $f \in X^*$, and

$$A\begin{pmatrix} \xi \\ x \end{pmatrix} = \begin{pmatrix} \eta\xi + f(x) \\ \xi z + Tx \end{pmatrix}. \tag{1.5}$$

Proof. Starting with (1.2), we get (1.4), where

$$\begin{pmatrix} \eta \\ z \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} f(x) \\ Tx \end{pmatrix} = A\begin{pmatrix} 0 \\ x \end{pmatrix}. \tag{1.6}$$

The formulas (1.6) define a canonical isomorphism from $L(E)$ onto the space of all matrices having the form (1.4). ∎

Let us note that equalities (1.1) and (1.5) just correspond to the standard rule of matrix multiplication. For example,

$$\eta\xi + f(x) = (\eta, f)\begin{pmatrix} \xi \\ x \end{pmatrix}.$$

It is important that the matrices corresponding to operators given by (1.4) [or (1.6)] will multiply under the usual rule. In more detail, if in addition to (1.4)

$$A_1 = \begin{pmatrix} \eta_1 & f_1 \\ z_1 & T_1 \end{pmatrix},$$

then

$$AA_1 = \begin{pmatrix} \eta\eta_1 + f(z_1) & \eta f_1 + T_1^* f \\ \eta_1 z + T z_1 & f_1 \otimes z + TT_1 \end{pmatrix},$$

where the tensor product $f_1 \otimes z$ means the linear operator in X acting by the formula $(f_1 \otimes z)(x) = f_1(x)z$.

We also note that if $A : E \to E$ is represented as the matrix (1.4), then

$$A^* = \begin{pmatrix} \eta & z \\ f & T^* \end{pmatrix},$$

where z is identified with its canonical image in X^{**}. This can be checked quite easily.

Since our main interest is in spectral theory, we pass to a consideration of the eigenvalues of an arbitrary matrix (1.4), which form the *spectrum* $\sigma(A)$ of the operator A. In general, this subset of the complex plane $\mathbb{C}$ coincides with the set of poles of the *resolvent* $R_\lambda(A) = (A - \lambda I)^{-1}$. (Here I is the identity operator.) The resolvent is a rational operator function on the complement subset $\rho(A) = \mathbb{C} \setminus \sigma(A)$, which is called the *resolvent set*. Certainly, in this context, the real space E must be complexified. This procedure preserves the canonical structure of one-dimensional extension, namely $E_{\mathbb{C}} = \mathbb{C} \oplus X_{\mathbb{C}}$, where $X_{\mathbb{C}} = X \oplus iX$, so $E_{\mathbb{C}}$ has a similar form. To calculate $R_\lambda(A)$ using (1.5), we also introduce the resolvent $R_\lambda(T)$ and denote it simply by R_λ for short. We will preserve the notation I for the identity operator in X.

Now let us construct the function

$$\Delta(\lambda) = \eta - \lambda - f(R_\lambda z). \tag{1.7}$$

It is a rational function whose poles belong to $\sigma(T)$. However, some points of $\sigma(T)$ may be regular for $\Delta(\lambda)$. A very simple example is $f = 0$ or $z = 0$; then $\Delta(\lambda) = \eta - \lambda$.

PROPOSITION 1.2 (cf. [1, pp. 28–29]). *Let $\lambda \in \rho(T)$ and $\Delta(\lambda) \neq 0$. Then $\lambda \in \rho(A)$ and*

$$R_\lambda(A) = \begin{pmatrix} \dfrac{1}{\Delta(\lambda)} & -\dfrac{1}{\Delta(\lambda)} R_\lambda^* f \\ -\dfrac{1}{\Delta(\lambda)} R_\lambda z & R_\lambda + \dfrac{1}{\Delta(\lambda)} R_\lambda^* f \otimes R_\lambda z \end{pmatrix}, \tag{1.8}$$

Proof. The product of the matrices

$$A - \lambda I = \begin{pmatrix} \eta - \lambda & f \\ z & T - \lambda I \end{pmatrix} \tag{1.9}$$

and

$$\begin{pmatrix} 1 & -R_\lambda^* f \\ -R_\lambda z & \Delta(\lambda) R_\lambda + R_\lambda^* f \otimes R_\lambda z \end{pmatrix} \tag{1.10}$$

is equal to

$$\begin{pmatrix} \Delta(\lambda) & -f(R_\lambda z) R_\lambda^* f + (R_\lambda^* f \otimes R_\lambda z)^* f \\ 0 & -(R_\lambda^* f) \otimes z + \Delta(\lambda) I + (T - \lambda I)(R_\lambda^* f \otimes R_\lambda z) \end{pmatrix}.$$

The latter is just $\Delta(\lambda)$I because of the general formulas $(\varphi \otimes e)^* f = f(e)\varphi$, $S(\varphi \otimes e) = \varphi \otimes Se$, where $\varphi \in X^*$, $e \in X$, $f \in X^*$, $S \in L(X)$. This yields that $\lambda \in \rho(A)$ under the condition $\Delta(\lambda) \neq 0$, and the matrix (1.10) is $\Delta(\lambda) R_\lambda(A)$. ∎

COROLLARY 1.3. *Let* $N = \{\lambda : \Delta(\lambda) = 0\}$. *Then*

$$N \subset \sigma(A) \subset N \cup \sigma(T). \tag{1.11}$$

Proof. By Proposition 1.2, $\rho(T) \cap \overline{N} \subset \rho(A)$, where the bar means the complement relative to $\mathbb{C}$. Therefore, $\overline{\rho(A)} \subset \overline{\rho(T)} \cup N$, i.e., $\sigma(A) \subset \sigma(T) \cup N$. On the other hand, if $\mu \in N$, we may apply (1.8) as $\lambda \to \mu$, since a sufficiently small neighborhood of μ does not contain any point $\lambda \in \sigma(T) \cup N$, $\lambda \neq \mu$. Looking at the northwest entry of the matrix (1.8), we see that $R_\lambda(A) \to \infty$ as $\lambda \to \mu$. This means that $\mu \in \sigma(A)$. ∎

In fact, one can explicitly calculate the determinant $D_A(\lambda) = \det(A - \lambda I)$ in terms of $\Delta(\lambda)$ and $D_T(\lambda) = \det(T - \lambda I)$.

PROPOSITION 1.4 (Cf. [3, 0.8.5]). *The following formula holds*:

$$D_A(\lambda) = \Delta(\lambda) D_T(\lambda). \tag{1.12}$$

Thus, we have the quotient

$$\Delta(\lambda) = \frac{D_A(\lambda)}{D_T(\lambda)}.$$

We call $\Delta(\lambda)$ the *quotient characteristic function of the extension A from* T. A more detailed notation is $\Delta_{A/T}(\lambda)$.

Proof. As we showed, the product of the matrices (1.9) and (1.10) is equal to $\Delta(\lambda)I$. But the matrix (1.10) yields

$$\begin{pmatrix} 1 & 0 \\ -R_\lambda z & \Delta(\lambda) R_\lambda \end{pmatrix}$$

on being multiplied by

$$\begin{pmatrix} 1 & (R_\lambda^*)f \\ 0 & I \end{pmatrix}$$

on the right side. Passing to determinants, we get

$$D_A(\lambda)\cdot\Delta^n(\lambda)\det R_\lambda = \Delta^{n+1}(\lambda).$$

It remains to note that $\det R_\lambda = D_T(\lambda)^{-1}$. ∎

COROLLARY 1.5. $\Delta_{A^*/T^*}(\lambda) = \Delta_{A/T}(\lambda)$.

The resolvent $R_\lambda = R_\lambda(T)$ may be written as $P_T(\lambda)/D_T(\lambda)$, where $P_T(\lambda)$ is an operator-valued polynomial of degree $n-1$. We have

$$\Delta(\lambda) = \frac{(\eta-\lambda)D_T(\lambda) - f(P_T(\lambda)z)}{D_T(\lambda)}$$

and

$$D_A(\lambda) = (\eta-\lambda)D_T(\lambda) - f(P_T(\lambda)z). \tag{1.13}$$

For example, if $f=0$ or $z=0$, then $\Delta(\lambda) = \eta-\lambda$ and

$$D_A(\lambda) = (\eta-\lambda)D_T(\lambda).$$

In this case $\sigma(A) = \{\eta\}\cup\sigma(T) = N\cup\sigma(T)$.

Another simple example is

$$A = \begin{pmatrix} \eta & f \\ z & I \end{pmatrix}.$$

In this case $\Delta(\lambda) = \eta - \lambda - f(z)/(1-\lambda)$. Suppose $f(z)\neq 0$. Then $N = \{\lambda\colon (\eta-\lambda)(1-\lambda) - f(z) = 0\}$. Since $D_T(\lambda) = (1-\lambda)^n$ $(T = I)$, we get, by (1.12), $D_A(\lambda) = (\eta-\lambda)(1-\lambda)^n - f(z)(1-\lambda)^{n-1}$. Obviously, $\sigma(T) = \{1\}$, $N\cap\sigma(T) = \varnothing$. If $n=1$ then $\sigma(A) = N$. If $n\geqslant 2$ then $\sigma(A) = N\cup\{1\} = N\cup\sigma(T)$.

Note that Corollary 1.3 follows from (1.12) in a purely algebraic way. However, the formula (1.8) is much more informative for applications. For example, (1.8) immediately implies the following

COROLLARY 1.6. *If μ is an m-multiple root of the quotient characteristic equation $\Delta(\lambda) = 0$, then μ is a pole of the resolvent $R_\lambda(A)$ of order at least m.*

Proof. The northwest entry of the matrix (1.8) has a pole $\lambda = \mu$ of order just m. ■

Recall that, in general, a multiple root of $D_A(\lambda)$ may appear as a simple pole of the resolvent $R_\lambda(A)$.

Under the condition of Corollary 1.6, there exists a Jordan chain of A of length m at the point $\lambda = \mu$. As a rule, such a chain can be constructed explicitly as indicated below.

Let us consider the rational vector function

$$u(\lambda) = \begin{pmatrix} 1 \\ -R_\lambda z \end{pmatrix}. \tag{1.14}$$

Obviously,

$$(A - \lambda I)u(\lambda) = \Delta(\lambda)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.15}$$

and by successive differentiations

$$(A - \lambda I)u^{(k)}(\lambda) - ku^{(k-1)}(\lambda) = \Delta^{(k)}(\lambda)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (k \geqslant 1) \tag{1.16}$$

for $\lambda \in \rho(T)$.

PROPOSITION 1.7. *Let μ be an m-multiple root of the equation $\Delta(\lambda) = 0$. Suppose that the function $R_\lambda z$ is regular at the point $\lambda = \mu$. Then the vectors*

$$u_k = \frac{u^{(k)}(\mu)}{k!} = \begin{pmatrix} \delta_{k0} \\ -R_\mu^{k+1} z \end{pmatrix} \qquad (0 \leqslant k \leqslant m - 1) \tag{1.17}$$

form a Jordan chain. (Here δ_{k0} is the Kronecker delta.)

Note that under the considered conditions the case $\mu \in \sigma(T)$ is not excluded. In this case the notation $R_\mu^{k+1} z$ means $\lim_{\lambda \to \mu} R_\lambda^{k+1} z$. This limit does exist by the assumption of regularity of $R_\lambda z$ at $\lambda = \mu$ (see below). From now on we will use this short notation.

Proof. The relation

$$(A - \mu I)u_k = u_{k-1} \qquad (1 \leqslant k \leqslant m - 1)$$

immediately follows from (1.16), and $(A - \mu I)u_0 = 0$ follows from (1.15). It remains to note that

$$\frac{1}{k!}\frac{d^k}{d\lambda^k}(R_\lambda z) = R_\lambda^{k+1} z. \qquad \blacksquare$$

By the way,

$$\Delta^{(k)}(\lambda) = -\delta_{k1} - k! f\left(R_\lambda^{k+1} z\right) \qquad (k \geqslant 1). \tag{1.18}$$

Hence, a root μ is simple iff $f(R_\mu^2 z) \neq -1$.

We call (1.17) the *special Jordan chain* at the point μ.

Consider the *maximal root subspace*

$$W = \bigcup_{k \geqslant 1} W^k \qquad \left[W^k = \ker(A - \mu I)^k\right]$$

at the point μ. The *order* p of the point μ is defined as the smallest k such that $W^k = W$. This coincides with the maximal order of Jordan blocks of A at the point μ and also with the order of the pole μ for $R_\lambda(A)$.

COROLLARY 1.8. *The multiplicity m of a root μ in $\Delta(\lambda) = 0$ does not exceed its order p.*

This inequality may be strict. For instance, if A is an $(n+1)$-dimensional Jordan block with the eigenvalue μ, then $\Delta(\lambda) = \mu - \lambda$, so $m = 1$, $p = n + 1$.

Let us note that

$$\dim W = m + q, \tag{1.19}$$

where q is the multiplicity of μ in $D_T(\lambda)$ [$q = 0$ if $D_T(\mu) \neq 0$]. Indeed, $\dim W$ coincides with the multiplicity of μ in $D_A(\lambda)$, which is $m + q$ by (1.12).

We see that $\dim W \geqslant m$, and $\dim W = m$ iff $\mu \overline{\in} \sigma(T)$. In this case the special Jordan chain $\{u_k\}_0^{m-1}$ is a basis in W, and hence the restriction $A \mid W$ is a single Jordan block.

2. SUBNORMED SPACES

The basic space X is called *subnormed* if it is provided with a *subnorm* π, which is, by our definition, a positive sublinear functional. In more detail, $\pi: X \to \mathbb{R}$ satisfies the following conditions:

(1) $\pi(x) > 0$ $(x \neq 0)$ and $\pi(0) = 0$ (*positivity*);
(2) $\pi(x_1 + x_2) \leqslant \pi(x_1) + \pi(x_2)$ (the *triangle inequality*);
(3) $\pi(\lambda x) = \lambda\pi(x)$ if $\lambda > 0$ (*positive homogeneity*).

Subnorms are standard in convex analysis (see, for instance, [6, Chapter 3]), but below we concentrate on the geometrical *structure* (X, π) similarly the standard structure of normed space.

A subnormed space X is said to be *strictly subnormed* if $\pi(x_1 + x_2) < \pi(x_1) + \pi(x_2)$ except when x_2 is a nonnegative multiple of x_1 or $x_1 = 0$.

Obviously, a subnorm π is a norm iff π is *symmetric*, i.e. $\pi(-x) = \pi(x)$. In any case, every subnorm is a convex function. For a subnorm π the function $\|x\|_\pi = \max(\pi(x), \pi(-x))$ is a norm. Since X is finite-dimensional, the linear topology defined by this norm coincides with the unique standard linear topology on X.

It follows from the triangle inequality that

$$-\pi(-x_2) \leqslant \pi(x_1 + x_2) - \pi(x_1) \leqslant \pi(x_2).$$

Therefore,

$$|\pi(x_1 + x_2) - \pi(x_1)| \leqslant \|x_2\|_\pi.$$

We conclude that every subnorm π is continuous. For this reason $\inf_{\|x\|_\pi = 1}[\pi(x)] > 0$, and we can introduce the *asymmetry coefficient*

$$\alpha = \sup_{x \neq 0} \frac{\pi(-x)}{\pi(x)} = \sup_{x \neq 0} \frac{\pi(x)}{\pi(-x)}. \tag{2.1}$$

Obviously, $1 \leqslant \alpha < \infty$, and $\alpha = 1$ iff π is symmetric.

By the above definitions

$$\alpha^{-1}\|x\|_\pi \leqslant \pi(x) \leqslant \|x\|_\pi, \tag{2.2}$$

and it turns out that the standard linear topology on X can be introduced by an arbitrary subnorm π using a fundamental system of neighborhoods of zero

$U_\varepsilon = \{x\colon \pi(x) < \varepsilon\}$, $\varepsilon > 0$. These neighborhoods are convex because of the convexity of π. They might be called the *open π-balls*, but we should remember that these "balls" may be not symmetric. Obviously, π is the Minkowski functional (or gauge) of the *closed unit π-ball* $U_1 = \{x\colon \pi(x) \leqslant 1\}$. Its boundary $S_1 = \{x\colon \pi(x) = 1\}$ is the *unit π-sphere*. Note that X is strictly subnormed iff S_1 does not contain any segment $x + \tau y$ $(0 \leqslant \tau \leqslant 1,\ y \neq 0)$.

Every subnorm π on X naturally generates at least three subnorms on the conjugate space X^*:

(1) $\pi^*(f) = \sup\limits_{x\neq 0} \dfrac{f(x)}{\pi(x)}$;

(2) $\pi^*_-(f) = \sup\limits_{x\neq 0} \dfrac{f(x)}{\pi(-x)} = \pi^*(-f)$;

(3) $æ^*(f) = \sup\limits_{x\neq 0} \dfrac{|f(x)|}{\pi(x)} = \sup\limits_{x\neq 0} \dfrac{|f(x)|}{\pi(-x)}$.

It is clear that none of these suprema changes if x is restricted to the corresponding closed unit π-ball or even unit π-sphere.

Obviously, $æ^*(f) = \max(\pi^*(f), \pi^*_-(f)) = \|f\|_{\pi^*}$. By definition, $f(x) \leqslant \pi^*(f)\pi(x)$; hence

$$\sup_{f\neq 0} \frac{f(x)}{\pi^*(f)} \leqslant \pi(x).$$

Actually,

$$\sup_{f\neq 0} \frac{f(x)}{\pi^*(f)} = \pi(x). \tag{2.3}$$

Since the unit π-ball is a convex body, for every x with $\pi(x) = 1$ there exists $f \in X^*$ such that $f(x) = 1$ and $\pi(y) \leqslant 1 \Rightarrow f(y) \leqslant 1$; this means that $\pi^*(f) = f(x) = 1$.

Passing to the conjugate space X^*, we may choose π^* as a canonical subnorm on X^*. Then we get (X^{**}, π) by repeating this construction, so the canonical isomorphism $X \to X^{**}$ is a π-isometry in a clear sense.

For every linear operator T in a subnormed space (X, π) one can define the value

$$\pi(T) = \sup_{x\neq 0} \frac{\pi(Tx)}{\pi(x)} = \sup_{\pi(x)=1} \pi(Tx) = \sup_{\pi(x)\leqslant 1} \pi(Tx).$$

In this way we obtain a subnormed space $(L(X), \pi)$. Moreover, it is a *subnormed algebra*:

$$\pi(T_1T_2) \leqslant \pi(T_1)\pi(T_2), \qquad \pi(I) = 1. \tag{2.4}$$

EXAMPLE 2.1. $\pi(-I) = \alpha$, where α is the asymmetry coefficient. Hence $\pi(-T) \leqslant \alpha\pi(T)$.

EXAMPLE 2.2. For any $e \in X$ and $\varphi \in X^*$

$$\begin{aligned}\pi(\varphi \otimes e) &= \sup_{x\neq 0} \frac{\pi(\varphi(x)e)}{\pi(x)} \\ &= \max\left(\pi(e) \sup_{\varphi(x)>0} \frac{\varphi(x)}{\pi(x)}, \pi(-e) \sup_{\varphi(x)<0} \frac{|\varphi(x)|}{\pi(x)}\right) \\ &\leqslant \max(\pi(e), \pi(-e)) \cdot \sup_{x\neq 0} \frac{|\varphi(x)|}{\pi(x)}.\end{aligned}$$

Thus,

$$\pi(\varphi \otimes e) \leqslant \|e\|_\pi \cdot \|\varphi\|_{\pi^*}. \tag{2.5}$$

The conjugate operator acts in (X^*, π^*), and the value $\pi^*(T^*)$ plays the same role as $\pi(T)$ above.

PROPOSITION 2.3. $\pi^*(T^*) = \pi(T)$.

Proof. By definition,

$$\pi^*(T^*) = \sup_{f\neq 0} \frac{\pi^*(T^*f)}{\pi^*(f)} = \sup_{f\neq 0,\, x\neq 0} \frac{f(Tx)}{\pi^*(f)\pi(x)} \leqslant \sup_{x\neq 0} \frac{\pi(Tx)}{\pi(x)}.$$

Thus, $\pi^*(T^*) \leqslant \pi(T)$. Replacing T by T^* and π by π^*, we obtain the opposite inequality. Therefore, the corresponding equality holds. ∎

Let us denote the asymmetry coefficient of the conjugate space (X, π^*) by α^*.

COROLLARY 2.4. $\alpha^* = \alpha$.

Proof. Set $T = -I$ in Proposition 2.3. ∎

In particular, Corollary 2.4 shows that π^* is a norm iff π is a norm.

3. NONNEGATIVE OPERATORS IN A SPACE WITH HYPERBOLIC CONE

Let (X, π) be a subnormed space. In this case the structure of the one-dimensional extension $E = \mathbb{R} \oplus X$ allows us to introduce a special construction, which we call a *hyperbolic cone*:

$$C = \left\{u \colon u = \begin{pmatrix} \xi \\ x \end{pmatrix},\ \xi \geqslant \pi(x)\right\}.$$

If π is a norm, the cone C is called *symmetric*.

From time to time such a construction has appeared in various contexts (for instance, see [6, Chapter 3, Section 15]; [7, Appendix, §3]). The symmetric case was more systematically considered by Fiedler and Haynsworth [2]. We elaborate a general case. In this way we obtain, in particular, a necessary and sufficient condition for a matrix (1.4) to define a C-nonnegative operator A.

Obviously, every hyperbolic cone C is convex and proper. The latter means that C is closed, pointed [i.e. $C \cap (-C) = \{0\}$], and solid (i.e. $\operatorname{Int} C \neq \varnothing$). In our case

$$\operatorname{Int} C = \left\{u \colon u = \begin{pmatrix} \xi \\ x \end{pmatrix},\ \xi > \pi(x)\right\}.$$

As usual, we write $u \geqslant 0$ if $u \in C$, and $u \gg 0$ if $u \in \operatorname{Int} C$.

Actually, every convex proper cone C in a $(n + 1)$-dimensional real space E can be considered as hyperbolic by a suitable choice of a decomposition $E = \mathbb{R} \oplus X$. In our further developments a hyperbolic cone is an *a priori* fixed geometric structure providing us with appropriate terms for the Perron-Frobenius theory (see Section 4).

A simplest example is the classical Lorentzian cone:

$$\xi \geqslant \left(\sum_{k=1}^{n} \xi_k^2 \right)^{1/2} \tag{3.1}$$

in $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$, where $\xi_1, \ldots, \xi_n$ are the canonical coordinates in $\mathbb{R}^n$. A more general construction is the l_p-cone

$$\xi \geqslant \left(\sum_{k=1}^{n} |\xi_k|^p \right)^{1/p} \tag{3.2}$$

with $p \geqslant 1$. It passes to

$$\xi \geqslant \max_{1 \leqslant k \leqslant n} |\xi_k| \tag{3.3}$$

as $p \to \infty$, so we can say that (3.3) is an l_∞*-cone*. By Proposition 3.1 the conjugate cone for the l_p-cone is the l_q-cone with $1/p + 1/q = 1$, including the limiting case $p = \infty$, $q = 1$.

In this series of examples the hyperbolic cones are symmetric. However, the coordinate cone $\mathbb{R}_+^{n+1}$ can be equipped with a nonsymmetric hyperbolic structure in the following way. Let

$$\xi = \frac{1}{n+1} \sum_{k=1}^{n+1} \xi_k, \qquad X = \{x\colon\ x \in \mathbb{R}^{n+1},\ \xi(x) = 0\}$$

and

$$\pi(x) = \max_{1 \leqslant k \leqslant n+1} [-\xi_k(x)] \qquad (x \in X).$$

Obviously, $\pi(x)$ is a subnorm on X. We have $\mathbb{R}^{n+1} = \mathbb{R}e \oplus X$, where e is the vector whose ξ_k's equal to 1. For any vector $u = \xi e + x$ $(x \in X)$ the inequality $\xi \geqslant \pi(x)$ means that all $\xi_k(u) \geqslant 0$, since $\xi_k(u) = \xi_k(x) + \xi$ $(1 \leqslant k \leqslant n+1)$.

There is no hyperbolic symmetric structure for R_+^{n+1} with $n \geqslant 2$, since every bounded complete cross section of this cone is an n-dimensional simplex, which is a body without central symmetry.

Let E be a space with a hyperbolic cone C. The conjugate cone $C^* \subset E^*$ is also hyperbolic in the following way.

PROPOSITION 3.1. $C^* = \{g\colon\ g = (\eta, f),\ \eta \geqslant \pi^*_-(f)\}$.

Proof. The property $g(u) \geqslant 0$ for all $u \geqslant 0$ can be written as

$$\xi \geqslant \pi(x) \quad \Rightarrow \quad \eta\xi + f(x) \geqslant 0. \tag{3.4}$$

If (3.4) is valid, then in particular $\eta\pi(x) + f(x) \geqslant 0$ for all $x \in X$. Replacing x by $-x$, we obtain

$$\eta \geqslant \sup_{x \neq 0} \frac{f(x)}{\pi(-x)} = \pi^*_-(f). \tag{3.5}$$

Conversely, if (3.5) is valid and $\xi \geqslant \pi(x)$, then we have

$$\eta\xi + f(x) \geqslant \pi^*_-(f)\pi(x) + f(x) \geqslant 0. \qquad \blacksquare$$

By the standard definition, the nonnegativity of an operator $A \in L(E)$ means that $AC \subset C$, and it is denoted by $A \geqslant 0$. How can we characterize the nonnegativity in terms of the matrix (1.4)? Some necessary conditions can be obtained immediately.

PROPOSITION 3.2. *If*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \geqslant 0,$$

then $\pi(z) \leqslant \eta$ *and* $\pi^*_-(f) \leqslant \eta$.

Proof. The first inequality is valid because

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \geqslant 0 \quad \text{and} \quad A \geqslant 0;$$

therefore

$$\begin{pmatrix} \eta \\ z \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} \geqslant 0.$$

The second one is the same for

$$A^* = \begin{pmatrix} \eta & z \\ f & T^* \end{pmatrix} \geqslant 0$$

(see Proposition 3.1). ∎

Now we establish a necessary and sufficient condition for an operator $A \in L(E)$ to be nonnegative.

THEOREM 3.3. *The property*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \geqslant 0$$

is equivalent to the inequality

$$\eta \geqslant \eta_0(T, z, f,) \equiv \sup_{\pi(x)=1} \left[\pi(z + Tx) - f(x) \right]. \tag{3.6}$$

Proof. The property $Au \geqslant 0$ for $u \geqslant 0$ can be written as

$$\xi \geqslant \pi(x) \quad \Rightarrow \quad \eta\xi + f(x) \geqslant \pi(\xi z + Tx). \tag{3.7}$$

If (3.7) is valid, then (3.6) follows for $\xi = \pi(x) = 1$.

Conversely, let (3.6) be valid. Then

$$\eta \geqslant \sup_{\pi(x) \leqslant 1} \left[\pi(z + Tx) - f(x) \right]$$

by the convexity of the function $\theta(x) = \pi(z + Tx) - f(x)$. This inequality means that

$$\eta + f(x) \geqslant \pi(z + Tx)$$

if $\pi(x) \leqslant 1$. Now for an arbitrary $x \in X$ and $\xi \geqslant \pi(x)$ we get

$$\eta\xi + f(x) = \xi\left[\eta + f\left(\frac{x}{\xi}\right)\right] \geqslant \xi\pi\left(z + T\left(\frac{x}{\xi}\right)\right) = \pi(\xi z + Tx)$$

if $\xi \neq 0$. The case $\xi = 0$ is trivial. ∎

Corollary 3.4. *Let*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix}.$$

If $A \geqslant 0$ *then*

$$\eta \geqslant \pi(T) - \pi(-z) - \pi^*(f), \tag{3.8}$$

and if

$$\eta \geqslant \pi(T) + \pi(z) + \pi^*(f), \tag{3.9}$$

then $A \geqslant 0$.

Proof. Both of the statements follow from Theorem 3.3 by the triangle inequality. ∎

Note that the second part of this corollary is an extension of Theorem 3 from [2].

Corollary 3.5. *If*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \geqslant 0$$

then $\pi(T) \leqslant (2\alpha + 1)\eta$, *where* α *is the asymmetry coefficient.*

Proof. $\pi(-z) + \pi^*(f) \leqslant \alpha\pi(z) + \alpha^*\pi^*(-f) = \alpha[\pi(z) + \pi_-^*(f)]$ $\leqslant 2\alpha\eta$ by Proposition 3.2. ∎

Corollary 3.6. *If*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \geqslant 0,$$

then $\eta \geqslant 0$, *and* $\eta = 0$ *iff* $A = 0$.

Remark 3.7. In the case $z = 0$, $f = 0$ a necessary and sufficient condition for $A \geqslant 0$ is $\eta \geqslant \pi(T)$. This follows from (3.6) immediately.

REMARK 3.8. In the symmetric case $\alpha = 1$, Corollary 3.5 only yields $\pi(T) \leqslant 3\eta$. In fact, this estimate can be sharpened via a more accurate elaboration of (3.6). Indeed, if $\pi(x) = 1$,

$$\begin{aligned}\pi(Tx) &\leqslant \tfrac{1}{2}[\pi(Tx+z) + \pi(Tx-z)] \\ &= \tfrac{1}{2}[\pi(Tx+z) + \pi(-Tx+z)] \\ &\leqslant \tfrac{1}{2}[f(x)+\eta] \\ &\quad + [f(-x)+\eta] = \eta,\end{aligned}$$

since $\pi(-x) = 1$ as well. Therefore, $\pi(T) \leqslant \eta$, or

$$\|T\| \leqslant \eta \tag{3.10}$$

in a more usual notation.

The function $\eta_0(T, z, f)$ on the space $Y = L(X) \oplus X \oplus X^*$ defined by (3.6) is a subnorm. Indeed, obviously, the triangle inequality holds and η_0 is positively homogeneous. Moreover, if $\eta_0(T, z, f) \leqslant 0$ then $\pi(Tx+z) - f(x) \leqslant 0$ for all x with $\pi(x) \leqslant 1$. In particular, we may take $x = 0$ and get $\pi(z) \leqslant 0$, so $z = 0$. Then $\pi(Tx) \leqslant f(x)$; therefore $Tx = 0$ on the hyperspace $f(x) \leqslant 0$. This yields $T = 0$, since the operator T is linear. Finally, $f = 0$, since $f(x) \geqslant 0$ for all $x \in X$.

Now the space $L(E) = \mathbb{R} \oplus Y$ can be equipped with the hyperbolic cone $\eta \geqslant \eta_0(T, z, f)$, which is exactly the cone of all nonnegative operators in X by Theorem 3.3. In such a way the hyperbolic cone structure canonically extends from E to the space of linear operators on E. It is important that η_0 may be not symmetric even in the case of symmetric π. For this reason the category of subnormed spaces (X, π) is preferable for the considered theory to its subcategory of normed spaces.

EXAMPLE 3.9. Let π be the Euclidean norm in the space $X = \mathbb{R}^n$,

$$\pi(x) = \|x\| = \left(\sum_{k=1}^{n} \xi_k^2\right)^{1/2}.$$

This norm is generated by the inner product

$$(x, x') = \sum_{k=1}^{n} \xi_k \xi_k',$$

and every linear functional on X has a form $f(x) = (x, x')$ with a suitable x'. Taking any vector z with $\|z\| = 1$, we set $f(x) = (x, z)$. Let us calculate

$$\eta_0(I, z, f) = \sup_{\|x\|=1} [\|x + z\| - (x, z)].$$

The function $\theta(x) = \|x + z\| - (x, z)$ on the unit sphere $\|x\| = 1$ is equal to $\sqrt{2 + 2\tau} - \tau$, where $\tau = (x, z)$ runs over the interval $|\tau| \leqslant 1$. Its maximum is equal to $\frac{3}{2}$, so $\eta_0(I, z, f) = \frac{3}{2}$ while

$$\eta_0(-I, -z, -f) = \sup_{\|x\|=1} [\|x + z\| + (x, z)] = \max_{|\tau| \leqslant 1} \left(\sqrt{2 + 2\tau} + \tau\right) = 3.$$

Our extension of Perron-Frobenius theory expounded below does not require any symmetry of π.

4. THE PERRON-FROBENIUS THEORY

We consider a linear operator $A \geqslant 0$ in a space $E = \mathbb{R} \oplus X$ with a hyperbolic cone

$$C = \left\{u \colon u = \begin{pmatrix} \xi \\ x \end{pmatrix}, \ \xi \geqslant \pi(x)\right\},$$

where π is a subnorm on X. Let $r = r(A)$ be the spectral radius of A.

By the classical Krein-Rutman theorem [5] there exists an eigenvector $u_0 \in C$, $Au_0 = ru_0$. Moreover, there exists a Jordan chain $\{u_k\}_0^{p-1}$, $u_0 \in C$, where p is the order of the point r (see [8, Theorem 5.2]). What more can we say using the hyperbolic structure? In other words, what are additional spectral properties of A in virtue of the condition (3.6)?

THEOREM 4.1. *Let $A \in L(E)$, where E is a space with a hyperbolic cone C, and let*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \geqslant 0.$$

Then:

(1) *The spectral radius $r = r(A)$ is a maximal nonnegative root of the quotient characteristic equation $\Delta(\lambda) = 0$.*

(2) *Let the multiplicity of the root r be m, and let the corresponding special Jordan chains be $\{u_k\}_0^{m-1}$, $\{g_k\}_0^{m-1}$ for A and A^* respectively. Then $u_0 \in C$, $g_0 \in C^*$, and the following duality relations hold:*

$$g_j(u_k) = 0 \quad (0 \leqslant k + j \leqslant m - 2),$$

$$g_j(u_k) = -\frac{\Delta^{(m)}(r)}{m!} > 0 \quad (k + j = m - 1).$$

(3) *The order of the point r is equal to m, so $\{u_k\}_0^{m-1}$ is the longest Jordan chain. This chain is a basis for the maximal root subspace W iff $r \overline{\in} \sigma(T)$.*

A key point of our proof of Theorem 4.1 is the following estimate for the resolvent $R_\lambda(A)$. [Recall that $R_\lambda \equiv R_\lambda(T)$.]

LEMMA 4.2. *Under the conditions of Theorem 4.1, the semiaxis $\lambda > r$ does not contain points $\lambda \in \sigma(T) \cup N$. On this set $\Delta(\lambda) < 0$, the functions $R_\lambda z$, $R_\lambda^* f$ are bounded, and*

$$\frac{c_1}{|\Delta(\lambda)|} \leqslant \|R_\lambda(A)\| \leqslant \frac{c_2}{|\Delta(\lambda)|}, \tag{4.1}$$

where $\|\cdot\|$ is any norm in $L(E)$, and c_1, c_2 are positive constants.

Proof. If $\lambda > r$, then $\lambda \overline{\in} N$ by Corollary 1.3, so the formula (1.8) is applicable under the additional assumption $\lambda \overline{\in} \sigma(T)$. It is a well-known fact that $-R_\lambda(A) \geqslant 0$ for $\lambda > r$. (This immediately follows from the Laurent expansion

$$-R_\lambda(A) = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}$$

and the condition $A \geqslant 0$.) Therefore $\Delta(\lambda) < 0$ by Corollary 3.6.

By Proposition 3.2 we obtain

$$\pi(-R_\lambda z) \leqslant 1, \qquad \pi^*(R_\lambda^* f) \leqslant 1, \tag{4.2}$$

whence by (2.2)

$$\|R_\lambda z\|_\pi \leqslant \alpha\pi(-R_\lambda z) \leqslant \alpha, \qquad \|R_\lambda^* f\|_{\pi^*} \leqslant \alpha^*\pi^*(R_\lambda^* f) \leqslant \alpha^* = \alpha, \tag{4.3}$$

so these functions are bounded.

Corollary 3.5 yields

$$\pi\left(-R_\lambda - \frac{1}{\Delta(\lambda)} R_\lambda^* f \otimes R_\lambda z\right) \leqslant \frac{2\alpha + 1}{|\Delta(\lambda)|};$$

hence

$$\|R_\lambda + \frac{1}{\Delta(\lambda)} R_\lambda^* f \otimes R_\lambda z\|_\pi \leqslant \frac{(2\alpha + 1)\beta}{|\Delta(\lambda)|}, \tag{4.4}$$

where β is the asymmetry coefficient of the operator space $L(X)$.

All norms in $L(E)$ are equivalent, so we may use one of them, say

$$\left\|\begin{pmatrix} \eta & f \\ z & T \end{pmatrix}\right\| = \max(|\eta|, \|z\|_\pi, \|f\|_\pi, \|T\|_\pi).$$

It follows from (4.3), (4.4), and (1.8) that

$$\frac{1}{|\Delta(\lambda)|} \leqslant \|R_\lambda(A)\| \leqslant \frac{(2\alpha + 1)\beta}{|\Delta(\lambda)|}. \tag{4.5}$$

It remains to show that if $\lambda_0 > r$ then $\lambda_0 \overline{\in} \sigma(T)$. We see from (4.3)–(4.4) that for $\lambda \overline{\in} \sigma(T)$

$$\|R_\lambda\|_\pi \leqslant \frac{c}{|\Delta(\lambda)|}, \qquad c = \text{const}. \tag{4.6}$$

Thus, $\overline{\lim}_{\lambda \to \lambda_0} \|R_\lambda\|_\pi < \infty$; hence $\lambda_0 \overline{\in} \sigma(T)$. ∎

REMARK 4.3. Let X be normed, $\pi(x) = \|x\|$. Using (3.10), it is easy to specify (4.6):

$$\|R_\lambda\| \leqslant \frac{2}{|\Delta(\lambda)|}.$$

The estimates (4.2) can be written as

$$\|R_\lambda z\| \leqslant 1, \qquad \|R_\lambda^* f\| \leqslant 1.$$

Proof of Theorem 4.1. (1): Since $r = r(A)$ is a point of the spectrum $\sigma(A)$, we have $\|R_\lambda(A)\| \to \infty$ as $\lambda \to r$, $\lambda > r$. Then $\Delta(\lambda) \to 0$ by (4.1); therefore $\Delta(r) = 0$. The root r is maximal nonnegative for $\Delta(\lambda)$, since $\Delta(\lambda) \neq 0$ for $\lambda > r$.

(2): The rational functions $R_\lambda z$ and $R_\lambda^* f$ are bounded for $\lambda > r$; hence, they are regular at $\lambda = r$. The special Jordan chains for A and A^* are

$$u_k = \begin{pmatrix} \delta_{k0} \\ -R_r^{k+1} z \end{pmatrix}, \quad g_k = \begin{pmatrix} \delta_{k0} \\ -(R_r^*)^{k+1} f \end{pmatrix} \qquad (0 \leqslant k \leqslant m-1).$$

Here $u_0 \in C$, $g_0 \in C^*$ by (4.2) and Proposition 3.1. Further,

$$g_j(u_k) = \delta_{k0}\delta_{j0} + f(R_r^{k+j+2} z) = \delta_{k+j,0} + f(R_r^{k+j+2} z).$$

The second summand can be calculated using (1.18), namely

$$f(R_r^{k+j+2} z) = -\frac{\delta_{k+j+1,1} + \Delta^{(k+j+1)}(r)}{(k+j+1)!}$$

However,

$$\delta_{k+j,0} - \frac{\delta_{k+j+1,1}}{(k+j+1)!} = 0$$

for all k, j. Thus,

$$g_j(u_k) = -\frac{\Delta^{(k+j+1)}(r)}{(k+j+1)!}.$$

It is just zero if $k+j \leqslant m-2$, but it is not zero if $k+j = m-1$. Moreover, $\Delta^{(m)}(r) < 0$ because of the Taylor expansion

$$\Delta(\lambda) = \frac{\Delta^{(m)}(r)}{m!}(\lambda - r)^m + \cdots$$

and the inequality $\Delta(\lambda) < 0$ for $\lambda > r$. Therefore $g_j(u_k) > 0$ for $k+j = m-1$.

(3): The order of the point r coincides with the order of this pole for $R_\lambda(A)$. It is equal to m by (4.1). Finally, as we know from Section 1, $\{u_k\}_0^{m-1}$ is a basis for W iff $r \overline{\in} \sigma(T)$. ∎

Let us formulate Theorem 4.1 in the case $m = 1$.

COROLLARY 4.4. *Under the hypotheses of Theorem* 4.1, *if the maximal nonnegative root r of the equation* $\Delta(\lambda) = 0$ *is simple, then*

$$u_0 = \begin{pmatrix} 1 \\ -R_\lambda z \end{pmatrix}, \qquad g_0 = \begin{pmatrix} 1 \\ -R_\lambda^* z \end{pmatrix}$$

is a Perron-Frobenius pair:

$$Au_0 = ru_0, \ A^*g_0 = rg_0, \qquad g_0(u_0) > 0$$

(*in fact*, $g_0(u_0) = -\Delta'(r)$).

The maximal root space W coincides with the eigenspace $W_1 = \ker(A - rI)$. *It is one-dimensional iff* $r \overline{\in} \sigma(T)$.

In the "degenerate" case $r \in \sigma(T)$ the eigenspace W_1 may have any dimension from 1 to $n+1$. For example, if $X = X_1 \oplus X_2$ and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & cI_2 \end{pmatrix}, \qquad |c| < 1,$$

where I_1 and I_2 are corresponding identity operators, then $A \geqslant 0$, $r = 1$, and $\dim W_1 = 1 + \dim X_1$; furthermore, $\Delta(\lambda) = 1 - \lambda$, so $r = 1$ is a simple root of $\Delta(\lambda)$.

REMARK 4.5. If $r \in \sigma(T)$, then the order of the point $\lambda = r$ for T does not exceed m. Indeed, $\lambda = r$ is a pole of order $\leqslant m$ for $R_\lambda = R_\lambda(T)$, as provided by the estimate (4.6). Coming back to Lemma 4.2, we note that all positive eigenvalues of the operator T lie on the interval $(0, r]$. In this connection one can introduce the "positive spectral radius"

$$r_+(T) = \max\{\lambda\colon \lambda \in \sigma(T),\ \lambda \geqslant 0\}.$$

[We set $r_+(T) = 0$ if T has no eigenvalues $\lambda \geqslant 0$.] As has been noted, we have

COROLLARY 4.6. *If*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \geqslant 0$$

then $r_+(T) \leqslant r(A)$.

As a consequence of this inequality we obtain that the above-introduced condition $r(A) \overline{\in} \sigma(T)$ is equivalent to the strong inequality $r_+(T) < r(A)$. However, it can occur that $r(A) < r(T)$.

EXAMPLE 4.7. Let $\dim X = 1$, $\pi(x) = |x|$. The matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

is nonnegative with respect to the cone $\xi \geqslant |x|$ in the two-dimensional space $E = \mathbb{R} \oplus X$. We have here

$$\Delta(\lambda) = 1 - \lambda - \frac{1}{1+\lambda} = \frac{\lambda^2}{1+\lambda}.$$

So $r(A) = 0$, though $\sigma(T) = \{-1\}$ and hence $r(T) = 1$ [and, certainly, $r_+(T) = 0$]. Note that $r(A)$ is a multiple root of $\Delta(\lambda)$ despite $r(A) \overline{\in} \sigma(T)$.

Now let us consider the special Jordan chain $\{u_k\}_0^{m-1}$ at the point r. All these vectors except u_0 lie outside the union $C \cup (-C)$, since their first coordinate is 0. May one reconstruct this chain so that all new vectors will be nonnegative? Generally speaking, the answer is no.

PROPOSITION 4.9. *Let X be a strictly subnormed space. If the spectral radius r of an operator*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \geqslant 0$$

has a multiplicity $m \geqslant 3$ as a root of $\Delta(\lambda) = 0$, then for the special Jordan chain $\{u_k\}_0^{m-1}$ at the point r every linear combination

$$u = \sum_{k=0}^{m-2} \gamma_k u_k$$

except $\gamma_0 u_0$ lies outside $C \cup (-C)$.

Proof. It is obvious if $\gamma_0 = 0$. Without loss of generality one can assume $\gamma_0 = 1$.

The equality $\Delta'(r) = 0$ yields $f(R_r^2 z) = -1$, i.e. $h(w) = 1$, where $h = R_r^* f$, $w = -R_r z$. Moreover, $\pi^*(h) \leqslant 1$, $\pi(w) \leqslant 1$ by (4.2). This is possible only if $\pi^*(h) = \pi(w) = 1$. In addition, by (1.18)

$$f(R_r^{k+1} z) = -\frac{\Delta^{(k)}(r)}{k!} = 0 \qquad (2 \leqslant k \leqslant m-1);$$

hence, $h(R_r^k z) = 0$ $(2 \leqslant k \leqslant m-1)$, so $h(v) = 0$, where

$$v = -\sum_{k=1}^{m-2} \gamma_k R_r^{k+1} z.$$

We get

$$\pi(w + \tau v) \geqslant h(w + \tau v) = h(w) = 1 \qquad (\tau \in \mathbb{R}). \tag{4.7}$$

Suppose that $u \in C \cup (-C)$. Then $u \in C$, since $\gamma_0 = 1$. This means that

$$\pi(w+v) \leqslant 1, \qquad \text{since} \quad u = \begin{pmatrix} 1 \\ w+v \end{pmatrix}.$$

Thus, $\pi(w+v) = 1$.

Let us consider the convex function $\theta(\tau) = \pi(w + \tau v)$ $(\tau \in \mathbb{R})$. Then $\theta(\tau) \geqslant 1$ and $\theta(0) = \theta(1) = 1$. Therefore $\theta(\tau) \equiv 1$ for $0 \leqslant \tau \leqslant 1$, i.e., the segment $w + \tau v$ $(0 \leqslant \tau \leqslant 1)$ lies on the unit π-sphere. This segment is reduced to a point, since X is strictly subnormed. Hence $w + \tau v$ does not depend on τ, so $v = 0$, i.e. $u = u_0$. ∎

COROLLARY 4.10. *Let X be strictly subnormed,*

$$A = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \geqslant 0,$$

and $r = r(A) \,\overline{\in}\, \sigma(T)$. If the multiplicity of r in $\Delta(\lambda)$ is $m \geqslant 3$, then at $\lambda = r$ there is no nonnegative root vector of order l, $2 \leqslant l \leqslant m - 1$.

Proof. Every such vector must be of the form

$$u = \sum_{k=0}^{l-1} \gamma_k u_k, \qquad \gamma_{l-1} \neq 0,$$

since $\dim W = m$ in the case $r \,\overline{\in}\, \sigma(T)$. This vector cannot be nonnegative, by Proposition 4.9. ∎

In particular, this corollary is applicable to any operator A preserving the Lorentzian cone or, more generally, the l_p-cone with $1 < p < \infty$. Certainly, in this way A should be given in the relevant matrix form.

REMARK 4.11. Stern and Wolkowicz [9] proved that the order p of the root r does not exceed 3 if C is Lorentzian. Therefore, in this case $m \leqslant 3$ by Theorem 4.1. Furthermore, it is proved in [9] that there exists only one Jordan block of length $\geqslant 2$ in the Lorentzian case with $p \geqslant 2$.

In Example 4.7 the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a root vector of order 2. We see that Corollary 4.10 cannot be extended to the case $m = 2$.

The requirement for X to be strictly subnormed is also essential. Indeed, the operator A, defined as a standard nilpotent Jordan block of order $n+1$ in the canonical coordinates of $\mathbb{R}^{n+1}$, is nonnegative with respect to the cone $\mathbb{R}_+^{n+1}$. Its root vectors $\{e_k\}_1^{n+1}$ (the canonical basis of $\mathbb{R}^{n+1}$) are nonnegative. It is easy to check that the condition $r(A) \overline{\in} \sigma(T)$ is fulfilled here for any decomposition $\mathbb{R}^{n+1} = \mathbb{R}e \oplus \mathbb{R}^n$ with $e \gg 0$.

In conclusion we apply Theorem 4.1 to clarify the asymptotic behavior of spectral radius $r(A)$ as $\eta \to +\infty$. Let us fix f, z, T in the matrix

$$A_\eta = \begin{pmatrix} \eta & f \\ z & T \end{pmatrix} \geqslant 0$$

and set $r(A_\eta) \equiv r_\eta$. It is a continuous function of η. Below $\eta_0 = \eta_0(T, z, f)$, so $\eta \geqslant \eta_0$ (see Theorem 3.3).

COROLLARY 4.12. *On the semiaxis $\eta \geqslant \eta_0$ the function r_η increases and satisfies the inequality*

$$-\min(\pi(z), \pi^*(-f)) \leqslant r_\eta - \eta \leqslant \min(\pi(-z), \pi^*(f)). \quad (4.8)$$

The following asymptotic relation holds:

$$r_\eta = \eta + \frac{f(z)}{\eta} + \frac{f(Tz)}{\eta^2} + O\left(\frac{1}{\eta^3}\right) \qquad (\eta \to \infty). \quad (4.9)$$

Proof. We have

$$\eta - r_\eta - f\left(R_{r_\eta} z\right) = 0 \qquad (\eta \geqslant \eta_0). \quad (4.10)$$

The inequality (4.8) immediately follows on taking into account the estimates (4.2). Moreover, (4.10) shows that if $r_{\eta_1} = r_{\eta_2}$ then $\eta_1 = \eta_2$, i.e., the function r_η is injective. Being continuous, it is monotone. It cannot be decreasing, by (4.10) at infinity. So it is increasing.

The identity (4.10) means that the function r_η $(\eta \geqslant \eta_0)$ is inverse to $\varphi(\lambda) = \lambda + f(R_\lambda z)$ $(\lambda \geqslant r_{\eta_0})$. The latter is a rational function, and its Laurent expansion at infinity is

$$\varphi(\lambda) = \lambda - \sum_{k=0}^{\infty} \frac{f(T^k z)}{\lambda^{k+1}}. \quad (4.11)$$

Its derivative at infinity is equal to 1, which is not zero. Therefore, $\varphi(\lambda)$ is invertible on a complex neighborhood of infinity, say, on $U = \{\lambda: |\lambda| > \rho\}$ with some $\rho > 0$. Let $\psi: \varphi(U) \to U$ be the inverse function. It is analytic, and its Laurent expansion at infinity has the form

$$\psi(\mu) = \mu + \sum_{k=0}^{\infty} \frac{a_k}{\mu^k}. \tag{4.12}$$

We obtain $a_0 = 0$, $a_1 = f(z)$, $a_2 = f(Tz), \dots$ from the identity $\varphi(\psi(\mu)) = \mu$ inserting (4.12) into (4.11). All coefficients are real, since the coefficients in (4.11) are so. Thus, for real η the function $\psi(\eta)$ is real-valued. That is true, in particular, on a semiaxis $\eta \geqslant \eta_1$ where (4.12) is valid. Being inverse to φ, $\varphi(\psi(\eta)) = \eta$, the function $\psi(\eta)$ coincides with r_η for $\eta \geqslant \eta_1$, and (4.9) follows. ∎

REMARK 4.13. To obtain explicitly the complete Laurent expansion of r_η one can use the Bürmann-Lagrange inversion formula (see [4, p. 138]).

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REFERENCES

1 Ben Noble and J. W. Daniel, *Applied Linear Algebra*, Prentice-Hall, 1977.
2 M. Fiedler and E. Haynsworth, Cones which are topheavy with respect to a norm, *Linear and Multilinear Algebra* 1:203–211 (1973).
3 R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge U.P., 1985.
4 A. Hurwitz and R. Courant, *Funktionentheorie*, Springer-Verlag, 1964.
5 M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in Banach space (in Russian), *Uspekhi Mat. Nauk 3* 1:3–95 (1948).
6 R. T. Rockafellar, *Convex Analysis*, Princeton U.P., 1970.
7 H. H. Schaefer, *Topological Vector Spaces*, Macmillan, 1966.
8 H. Schneider, Geometric conditions for the existence of positive eigenvalues of matrices, *Linear Algebra Appl.* 38:253–271 (1981).
9 R. J. Stern and H. Wolkowicz, Invariant ellipsoidal cones, *Linear Algebra Appl.* 150:81–106 (1991).

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